MATH 504 HOMEWORK 1

Due Monday, January 25.

Problem 1. *Suppose that a and b are two sets. Use the axioms to show that their symmetric difference, $a \triangle b$, is also a set.*

Problem 2. *Let y be a set of ordinals.*

(1) *If y is nonempty, show that the $\in$-minimal element in y is unique.*
(2) *Show that $\bigcup y$ is an ordinal.*

For the following assume that $\alpha, \beta, \gamma, \delta, \xi$ are ordinals.

Problem 3. *Show that $\alpha < \beta$ implies that $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example to show that $\leq$ cannot be replaced with $<$. Also, show that*

$$\alpha \leq \beta \to (\exists! \delta)(\alpha + \delta = \beta).$$

Problem 4. *Show that if $\gamma > 0$, then $\alpha < \beta$ implies that $\gamma \cdot \alpha < \gamma \cdot \beta$ and $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Give an example to show that $\leq$ cannot be replaced with $<$. Also, show that*

$$(\alpha \leq \beta \wedge \alpha > 0) \to (\exists! \delta, \xi)(\xi < \alpha \wedge \alpha \cdot \delta + \xi = \beta).$$

Problem 5. *Verify that ordinal exponentiation satisfies $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$ and $(\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}$.*

Problem 6. *Show in ZF^{-} (i.e. the ZF axioms minus Foundation) that for any set X the following are equivalent:*
(a) X can be well ordered,
(b) There is a $C : (\mathcal{P}(X) \setminus \{0\}) \to X$ such that $\forall Y \subset X (Y \neq \emptyset \to C(Y) \in Y)$.

Hint for (b)$\to$ (a): Fix $p \neq X$, and let $C(Y) = p$ if $Y \notin \mathcal{P}(X) \setminus \{0\}$. Define by transfinite recursion,

$$F(\alpha) = C(X \setminus \{F(\xi) \mid \xi < \alpha\}).$$